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Unique physical characterisation of Haag–Ruelle scattering states

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Abstract. A physically evident requirement on asymptotic product states is formulated in a mathematically precise way and shown to fix the S-matrix uniquely for relativistic field theories of short-range interactions.

1. Introduction

Let $\mathscr{H}$ be the Hilbert space of a given quantum theory. The basic problem of fixing a corresponding scattering theory is well known to be the identification of states describing certain asymptotic physical situations for large positive (or negative) times. It is generally believed that the asymptotic scattering configurations of pure states may be indexed by vectors Φ^0 of the Hilbert space $\mathscr{H}^0$ of a suitable (pseudo)*free* theory. More precisely, one usually assumes existence of two isometric mappings V_{out}, V_{in} from $\mathscr{H}^0$ into $\mathscr{H}$ such that $V_{\text{out}}\Phi^0$ (respectively $V_{\text{in}}\Phi^0$) is a Heisenberg state vector describing a physical situation essentially the same for large positive (respectively negative) times as that indexed by Φ^0. Thus, given a system in a state corresponding to the incoming configuration indexed by the unit vector Φ^0, the probability for detecting it in a state corresponding to the outgoing configuration indexed by the unit vector Ψ^0 is

$$W(\Phi^0 \to \Psi^0) = |\langle V_{\text{out}}\Psi^0 | V_{\text{in}}\Phi^0\rangle|^2$$

where $\langle \ | \ \rangle$ denotes the inner product of $\mathscr{H}$. The S-matrix operator in the Heisenberg picture is $S \equiv V_{\text{in}} V_{\text{out}}^{-1}$, hence

$$\langle V_{\text{out}}\Psi^0 | V_{\text{in}}\Phi^0\rangle = \langle V_{\text{out}}\Psi^0 | S V_{\text{out}}\Phi^0\rangle$$

for all $\Phi^0, \Psi^0 \in \mathscr{H}^0$.

The problem is how to specify physically the isometric operators V_{out} and V_{in}.

In the Haag–Ruelle scattering theory (see Reed and Simon 1979, and references therein) isometric operators V_{out} and V_{in} are specified mathematically, satisfying the additional requirement of relativistic covariance (Streater 1967):

$$U(\Lambda, a) V_{\text{out (in)}} = V_{\text{out (in)}} U^0(\Lambda, a) \tag{1}$$

where $U(\Lambda, a)$ and $U^0(\Lambda, a)$ are (strongly) continuous unitary representations of the restricted Poincaré group $\mathscr{P}_+^\uparrow$ in $\mathscr{H}$ and $\mathscr{H}^0$ respectively, if we consider only Bose

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fields. However, no stringent physical justification for this identification of scattering states is given.

The main purpose of the present paper is to fill this gap. Introducing the notion of asymptotic localisation† of operator sequences, we shall be able to formulate precisely a natural physical criterion fixing the choice of $V_{\text{out (in)}}$ uniquely. Of course, the correct choice will turn out to be that made by Haag (1958).

The techniques used here may be considered as a systematic development of those introduced by Hepp (1964, 1965) for the analysis of *non-overlapping* scattering states.

2. General framework and basic strategy

For convenience, let us consider the simple case where the free field theory describing the scattering configurations is that of stable neutral scalar particles of a single type with (physical) mass $m > 0$. Then $\mathcal{H}^0$ is the corresponding Fock space with the usual‡ associative symmetric tensor product $\otimes_s$:

$$\mathcal{H}^0 \equiv \mathcal{H}^0_0 \oplus \mathcal{H}^0_1 \oplus \mathcal{H}^0_2 \oplus \dots$$

where

$$\mathcal{H}^0_n \equiv \begin{cases} \mathbb{C} & \text{for } n = 0 \\ L^2(\mathbb{R}^3) & \text{for } n = 1 \\ \mathcal{H}^0_1 \otimes_s \dots \otimes_s \mathcal{H}^0_1 & \text{for } n > 1 \end{cases}$$

denotes the free n-particle subspace.

We do not require asymptotic completeness but assume the physical spectrum and mass-gap condition that would be implied by asymptotic completeness:

$$\mathcal{H} = E(\{0\} \cup M_m \cup V_m)\mathcal{H} \tag{2}$$

where E denotes the unique projector-valued measure on $\mathbb{R}^4$ with§

$$U(1, a) = \int \mathrm{d}E(p) \exp(\mathrm{i}pa). \tag{3}$$

By M_m we denote the one-particle mass shell

$$\{p \in \mathbb{R}^4 : p^2 = m^2, p^0 > 0\}$$

and by V_m the free multi-particle spectrum

$$\{p \in \mathbb{R}^4 : p^2 \geqslant 4m^2, p^0 > 0\}.$$

Obviously, without loss of generality, we may assume

$$\mathcal{H}^0_0 = E(\{0\})\mathcal{H} \qquad \mathcal{H}^0_1 = E(M_m)\mathcal{H} \tag{4}$$

and

$$U(\Lambda, a)\Phi = U^0(\Lambda, a)\Phi \qquad \text{for } \Phi \in \mathcal{H}^0_0 \oplus \mathcal{H}^0_1. \tag{5}$$

† Perhaps one should point out that the localisation properties exploited in the present paper, in contrast to those studied by Haag and Swieca (1965), are certainly not suited for an investigation of asymptotic completeness in quantum field theory.

‡ For instance, if Ω_0 denotes the free vacuum state vector and $A_0^{(-)}(\varphi_j)$ the negative-frequency part of the free scalar field smeared by the test function φ_j, then $(A_0^{(-)}(\varphi_1)\Omega_0) \otimes_s \dots \otimes_s (A_0^{(-)}(\varphi_n)\Omega_0) = A_0^{(-)}(\varphi_1) \dots A_0^{(-)}(\varphi_n)\Omega_0$ and $\Omega_0 \otimes_s \Phi^0 = \Phi^0 \otimes_s \Omega_0 = \Phi^0$ for all $\Phi^0 \in \mathcal{H}^0$.

§ We use natural units in which $c = \hbar = 1$.

Then V_{out} and V_{in} are fixed on $\mathcal{H}_0^0$ and $\mathcal{H}_1^0$ up to an unessential phase factor. Let us take the natural choice

$$V_{\text{out (in)}}\Phi = \Phi \qquad \text{for } \Phi \in \mathcal{H}_0^0 \oplus \mathcal{H}_1^0. \tag{6}$$

Thus, due to isometry, the whole problem reduces to inductive determination of

$$(V_{\text{out (in)}}\Phi^0) \otimes_\pm (V_{\text{out (in)}}\Psi^0) \equiv V_{\text{out (in)}}(\Phi^0 \otimes_s \Psi^0) \tag{7}$$

for suitable $\Phi^0, \Psi^0 \in \mathcal{H}_1^0, \mathcal{H}_2^0, \mathcal{H}_3^0, \ldots$. This means that we have to determine the products $\otimes_\pm$ for a sufficiently large class of states from physically evident properties, which have to be formulated in a mathematically precise way.

Now, $\otimes_+$ and V_{out} (respectively $\otimes_-$ and V_{in}) are defined such that $(V_{\text{out}}\Phi_1^0) \otimes_+ (V_{\text{out}}\Phi_2^0)$ (respectively $(V_{\text{in}}\Phi_1^0) \otimes_- (V_{\text{in}}\Phi_2^0)$) describes a physical situation essentially the same for large positive (respectively negative) times as that indexed by the free Fock vector $\Phi_1^0 \otimes_s \Phi_2^0$. Therefore, the relevant physical properties of $\otimes_+$ (respectively $\otimes_-$) may be read off from those of the free product $\otimes_s$ (cf Froissart and Taylor 1967). In order to be more explicit let Φ_j^0 be a free n_j-particle state, well localised at time zero ($j = 1, 2$). Denote by K_j the velocity cone of Φ_j^0, i.e. the closure of the set of all four-vectors of the form $(t, t\boldsymbol{v})$, where t is an arbitrary time and $\boldsymbol{v}$ a three-velocity allowed for at least one of the *free* particles described by Φ_j^0. Obviously, with increasing (Euclidean) distance from the origin the probability of finding any of the particles outside K_j will rapidly tend to zero (at least, if the probability distribution for the momenta is smooth). In other words, the physical situation described by Φ_j^0 far from the origin outside K_j is essentially that of the vacuum. Recall that $\Phi_1^0 \otimes_s \Phi_2^0$ describes the uncorrelated joint presence of both systems corresponding to Φ_1^0 and Φ_2^0. Therefore, if Φ_1^0 and Φ_2^0 are asymptotically *non-overlapping*, i.e. if $K_1 \cap K_2 = \{0\}$, the asymptotic physical situation described by $\Phi_1^0 \otimes_s \Phi_2^0$ is the following: with increasing (Euclidean) distance from the origin the physical situation outside K_1 (respectively K_2) approaches rapidly that described by Φ_2^0 (respectively Φ_1^0).

This has obvious consequences for the expectation values of local measurements: Let $\mathcal{O}$ be a space–time region for which it is impossible to send a signal from $\mathcal{O}$ to K_2 and back again to $\mathcal{O}$. Then due to Einstein causality, measurements within $\mathcal{O}$ depend only on the physical situation outside K_2. Consequently, if $\mathcal{O}$ is very far from the origin, measurements within $\mathcal{O}$ should yield essentially the same results for $\Phi_1^0 \otimes_s \Phi_2^0$ and Φ_1^0. Of course, the measurements within $\mathcal{O}$ must not amplify the weak influence of Φ_0^2 outside K_2 too much.

For $(V_{\text{out (in)}}\Phi_1^0) \otimes_\pm (V_{\text{out (in)}}\Phi_2^0)$, describing the same asymptotic physical situation for time $t \to \pm\infty$ as $\Phi_1^0 \otimes_s \Phi_2^0$, these considerations show the following: Let $\mathcal{O}$ be a space–time region as considered above with the additional property $\pm x^0 > 0$ for $x \in \mathcal{O}$. Then $\mathcal{O}_\lambda \equiv \lambda\mathcal{O}$ is moving into the future (respectively past) very far from K_2 for $\lambda \to +\infty$. Therefore the expectation values for a sequence of self-adjoint operators A_λ in $\mathcal{H}$ are the same for $(V_{\text{out (in)}}\Phi_1^0) \otimes_\pm (V_{\text{out (in)}}\Phi_2^0)$ and $V_{\text{out (in)}}\Phi_1^0$ in the limit $\lambda \to \infty$, i.e. we have the asymptotic condition

$$\lim_{\lambda \to +\infty} [\langle (V_{\text{out (in)}}\Phi_1^0) \otimes_\pm (V_{\text{out (in)}}\Phi_2^0) | A_\lambda ((V_{\text{out (in)}}\Phi_1^0) \otimes_\pm (V_{\text{out (in)}}\Phi_2^0))\rangle$$

$$- \langle V_{\text{out (in)}}\Phi_1^0 | A_\lambda V_{\text{out (in)}}\Phi_1^0 \rangle \langle V_{\text{out (in)}}\Phi_2^0 | V_{\text{out (in)}}\Phi_2^0 \rangle] = 0$$

provided that the following three conditions are fulfilled.

(i) $(V_{\text{out (in)}}\Phi_1^0) \otimes_\pm (V_{\text{out (in)}}\Phi_1^0)$ is in the domain of A_λ for all λ.

(ii) A_λ corresponds to a measurement which can essentially be performed within $\mathcal{O}_\lambda$ in the limit $\lambda \to \infty$.

(iii) Small changes of the physical situation inside the causal completion of $\mathcal{O}_\lambda$ are not amplified too much in the limit $\lambda \to \infty$.

While it is physically evident that the above asymptotic condition should hold, it is not at all obvious how it determines the product $\otimes_{+(-)}$. How this works is the main problem of the present paper and will be considered in § 5. The solution depends crucially on very special properties, called K-approachability, of a sufficiently large class of scattering states. These properties are analysed in § 4. In order to make the above asymptotic condition precise, we have to specify sequences of observables A_λ, said to be asymptotically localised in $\mathcal{O}_\lambda$, for which conditions (i)–(iii) are expected to hold. In order to yield (in § 3) a notion of asymptotic localisation which undoubtedly serves the purpose outlined above, let us assume the theory with interaction to be of Wightman type (Streater and Wightman 1964). Thus we assume the existence of a quantum field $A(x)$ and a dense linear subset D of $\mathcal{H}$ fulfilling the following requirements.

W1. $\mathcal{H}_0^0 \subset D$.

W2. $U(\Lambda, a)D \subset D$ for all $(\Lambda, a) \in \mathcal{P}_+^\uparrow$.

W3. For every tempered test function† $\varphi \in \mathcal{S}(\mathbb{R}^4)$ the 'smeared field operator' $A(\varphi) \equiv \int dx\, A(x)\varphi(x)$ is given as a linear (unbounded) operator in $\mathcal{H}$, defined on D.

W4. $A(\varphi)D \subset D$ for all $\varphi \in \mathcal{S}(\mathbb{R}^4)$.

W5. $A(\varphi^*) \subset A(\varphi)^*$ for all $\varphi \in \mathcal{S}(\mathbb{R}^4)$.

W6. The formal expectation values $\langle \Phi | A(x) \Phi \rangle$ are tempered Schwartz distributions for $\Phi \in D$; i.e. $\varphi \to \langle \Phi | A(\varphi) \Phi \rangle$ is a continuous linear mapping from $\mathcal{S}(\mathbb{R}^4)$ into $\mathbb{C}$.

W7. $U(\Lambda, a)A(x)U(\Lambda, a)^{-1} = A(\Lambda x + a)$ in the sense of distributions, for all $(\Lambda, a) \in \mathcal{P}_+^\uparrow$.

W8. $[A(x), A(y)]_- \equiv A(x)A(y) - A(y)A(x) = 0$ in the sense of distributions, if‡ $x \times y$.

Finally, let us choose a unit *vacuum* vector $\Omega \in \mathcal{H}_0^0$ and assume Ω to be cyclic with respect to the set of all smeared field operators.

3. Asymptotic localisation

By $\hat{\mathcal{P}}(\mathcal{O})$, $\mathcal{O}$ an open subset of $\mathbb{R}^4$, let us denote the smallest * algebra with unit§ $1 \restriction D$ containing all operators of the form

$$\sum_{j=1}^{n} \int dx_1 \ldots dx_j\, A(x_1) \ldots A(x_j)\varphi_j(x_1, \ldots, x_j)$$

where the $\varphi_j \in \mathcal{S}(\mathbb{R}^{4j})$ vanish outside $\mathcal{O} \times \ldots \times \mathcal{O}$, and n is an arbitrary positive integer. Observables corresponding to (essentially self-adjoint) elements of $\hat{\mathcal{P}}(\mathcal{O})$ are usually considered to be measurable within the space–time region $\mathcal{O}$. In the spirit of this interpretation we introduce the following definitions.

† We use Schwartz's (1966) notation for test function spaces.

‡ We write $x \times y$ for the statement that x is space-like with respect to y. Accordingly, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are subsets of $\mathbb{R}^4$, we write $\mathcal{O}_1 \times \mathcal{O}_2$ for the statement that $x \times y$ for every pair $(x, y) \in \mathcal{O}_1 \times \mathcal{O}_2$.

§ As usual, we denote by $A \restriction D$ the restriction of the operator A to the domain D.

Definition 1. Let $\{\mathcal{O}_\lambda\}_{\lambda>0}$ be a sequence of open subsets of $\mathbb{R}^n$ and let $\{\varphi_\lambda\}_{\lambda>0}$ be a sequence of test functions in $\mathscr{S}(\mathbb{R}^n)$. Then we say that φ_λ is *asymptotically localised* in $\mathcal{O}_\lambda$, iff the following two conditions are fulfilled.

(i) All Schwartz norms of φ_λ are bounded polynomially in λ; i.e. for every $n_1 \in \mathbb{N} \equiv \{1, 2, 3, \ldots\}$ there is a $n_2 \in \mathbb{N}$ such that

$$|\varphi_\lambda|_{n_1,\mathbb{R}^n} < n_2 + \lambda^{n_2} \qquad \text{for all } \lambda > 0$$

where we use the notation

$$|\varphi|_{k,M} \equiv \sup_{\chi\in M}(1+\|\chi\|)^k \max_{\substack{\alpha^1,\ldots,\alpha^n\in\{0\}\cup\mathbb{N}\\ \alpha^1+\ldots+\alpha^n<k}} \left|\left(\frac{\partial}{\partial\chi^1}\right)^{\alpha^1}\cdots\left(\frac{\partial}{\partial\chi^n}\right)^{\alpha^n}\varphi(\chi)\right|$$

where $\|\chi\| \equiv [(\chi^1)^2 + \ldots + (\chi^n)^2]^{1/2}$, for arbitrary $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $M \subset \mathbb{R}^n$.

(ii) $\sup_{\lambda>0}(1+\lambda)^{n_1}|\varphi_\lambda|_{n_2,\mathbb{R}^4\setminus\mathcal{O}_\lambda} < \infty$ for all $n_1, n_2 \in \mathbb{N}$.

An important example for definition 1 is given by

Lemma 1. Let $\varphi \in \mathscr{S}(\mathbb{R}^4)$ and let f be a smooth positive-frequency solution of the Klein–Gordon equation; i.e.

$$f(x) = (2\pi)^{-3/2}\int_{p^0=(\boldsymbol{p}^2+m^2)^{1/2}} \frac{\mathrm{d}\boldsymbol{p}}{2p^0}\hat{f}(\boldsymbol{p})\exp(-\mathrm{i}px) \tag{8}$$

with $\hat{f} \in \mathscr{S}(\mathbb{R}^3)$. Then, for every $\varepsilon > 0$, the sequence of test functions

$$\varphi_\lambda^{+(-)}(x) \equiv \int_{y^0=\pm\lambda} \mathrm{d}\boldsymbol{y}\,\varphi(x-y)f(y) \tag{9}$$

is asymptotically localised in† $\mathcal{O}_\lambda^\pm \equiv U_{\varepsilon\lambda}(\{x \in K_f\colon x^0 = \pm\lambda\})$, where K_f denotes the velocity cone of f; i.e.

$$K_f \equiv \overline{\{pt\colon \hat{f}(\boldsymbol{p}) \neq 0, p^0 = (\boldsymbol{p}^2+m^2)^{1/2}, t \in \mathbb{R}^1\}}.$$

Proof. Note that the inequality

$$\left(1 + \inf_{x\in M}\|x-\eta\|\right)^j |\varphi(\cdot-\eta)|_{k,M} \leqslant (1+\|\eta\|)^k|\varphi|_{j+k,M-\eta}$$

holds for arbitrary $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $\eta \in \mathbb{R}^n$ and $j, k \in \mathbb{N}$. If this is applied to (9) it yields

$$|\varphi_\lambda^\pm|_{k,\mathbb{R}^4\setminus\mathcal{O}_\lambda} \leqslant |\varphi|_{j+k,\mathbb{R}^4}\int_{y^0=\pm\lambda}\mathrm{d}\boldsymbol{y}\,\frac{(1+\|y\|)^k|f(y)|}{(1+\inf_{x\in\mathbb{R}^4\setminus\mathcal{O}_\lambda}\|x-y\|)^j}$$

† We use standard notation. For example, let $\lambda > 0$, $G \subset \mathbb{R}^4$, $G' \subset \mathbb{R}^4$, $(\Lambda, a) \in \mathscr{P}_+^\uparrow$. Then

$$U_\lambda(G) \equiv \{x \in \mathbb{R}^4\colon \|x-x'\| < \lambda \text{ for some } x' \in G\}$$

$$\bar{G} \equiv \text{closure of } G$$

$$\mathbb{R}^4\setminus G \equiv \{x \in \mathbb{R}^4\colon x \notin G\}$$

$$\lambda G \equiv \{x \in \mathbb{R}^4\colon x = \lambda x' \text{ for some } x' \in G\}$$

$$\Lambda G \pm a \equiv \{x \in \mathbb{R}^4\colon x = \Lambda x' \pm a \text{ for some } x' \in G\}$$

$$G \pm G' \equiv \bigcup_{a\in G'}(G \pm a).$$

for every $\mathcal{O}_\lambda \subset \mathbb{R}^4$ and all $j, k \in \mathbb{N}$. Therefore, since

$$\sup_{x \in \mathbb{R}^4 \setminus K_f} \|x\|^n |f(x)| < \infty$$

holds for all $n \in \mathbb{N}$ (Ruelle 1962, Lücke 1974), there is a rapidly decreasing sequence of complex numbers c_λ fulfilling the inequality·

$$|\varphi_\lambda^\pm|_{k,\mathbb{R}^4 \setminus \mathcal{O}_\lambda} \leqslant c_\lambda + |\varphi|_{j+k,\mathbb{R}^4} \max_{x \in \mathbb{R}^n} |f(x)| \int_{\substack{y^0 = \pm\lambda \\ y \in K_f}} \mathrm{d}y \frac{(1+\|y\|)^k}{(1+\inf_{x \in \mathbb{R}^4 \setminus \mathcal{O}_\lambda} \|x-y\|)^j}$$

for all $\lambda > 0$. Thus, choosing $\mathcal{O}_\lambda = \varnothing$, we see that condition (i) of definition 1 is fulfilled. On the other hand, choosing $\mathcal{O}_\lambda = \mathcal{O}_\lambda^\pm$ and j large enough, we see also condition (ii) to be fulfilled.

Definition 2. Let $\{\mathcal{O}_\lambda\}_{\lambda > 0}$ be a sequence of open subsets of $\mathbb{R}^4$ and let $\{B_\lambda\}_{\lambda > 0}$ be a sequence of $\hat{\mathcal{P}}(\mathbb{R}^4)$ operators. Then we say that B_λ is *asymptotically localised* in $\mathcal{O}_\lambda$, iff there are complex numbers c_λ, test functions $\varphi_{j,\lambda}$ and a positive integer n such that the following three conditions are fulfilled.

(i)

$$B_\lambda = c_\lambda + \sum_{j=1}^{n} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j A(x_1) \ldots A(x_j) \varphi_{j,\lambda}(x_1, \ldots, x_j)$$

for all $\lambda > n$.

(ii) $|c_\lambda| < n + \lambda^n$ for all $\lambda > n$.

(iii) For every fixed j, the test functions $\varphi_{j,\lambda}$ are asymptotically localised in $\mathcal{O}_\lambda \times \ldots \times \mathcal{O}_\lambda \subset \mathbb{R}^{4j}$.

Definition 2 is intended to be such that, if B_λ is asymptotically localised in $\mathcal{O}_\lambda$, then, for large $\lambda > 0$, we may consider B_λ to be essentially like an element of $\hat{\mathcal{P}}(\mathcal{O}_\lambda)$. What this precisely means is the content of the following simple lemma that we state without proof.

Lemma 2. Let $\varepsilon > 0$ and let $B_{j,\lambda}$ be asymptotically localised in $\mathcal{O}_{j,\lambda}, j = 1, \ldots, n$. Then there are sequences of operators $B'_{j,\lambda} \in \hat{\mathcal{P}}(U_\varepsilon(\mathcal{O}_{j,\lambda}))$ such that† $B_{1,\lambda} \ldots B_{n,\lambda}\Phi \underset{\lambda \to \infty}{\approx} B'_{1,\lambda} \ldots B'_{n,\lambda}\Phi$ for all $\Phi \in D$. Here, for each j with $B_{j,\lambda} \in \hat{\mathcal{P}}(\mathcal{O}_{j,\lambda})$ we may choose $B'_{j,\lambda} = B_{j,\lambda}$.

W8 is well known to imply that the elements of $\hat{\mathcal{P}}(\mathcal{O})$ commute with those of $\hat{\mathcal{P}}(\mathcal{O}')$, if $\mathcal{O} \times \mathcal{O}'$. Thus, we have the following corollary.

Corollary. Let $\varepsilon > 0$ and let $B_{j,\lambda}$ be asymptotically localised in $\mathcal{O}_{j,\lambda}$ for $j = 1, \ldots, 4$. If $\mathcal{O}_{2,\lambda} \times U_{\varepsilon\lambda}(\mathcal{O}_{3,\lambda})$ for all $\lambda > 0$, then

$$B_{1,\lambda}B_{2,\lambda}B_{3,\lambda}B_{4,\lambda}\Phi \underset{\lambda \to \infty}{\approx} B_{1,\lambda}B_{3,\lambda}B_{2,\lambda}B_{4,\lambda}\Phi$$

for all $\Phi \in D$.

One can easily prove several useful properties of the notion of asymptotic localisation. We list only a few.

† If $\{\Phi_\lambda\}_{\lambda > 0}$ and $\{\Phi'_\lambda\}_{\lambda > 0}$ are sequences of vectors in a Hilbert space with norm $|.|$, we write $\Phi_\lambda \underset{\lambda \to \infty}{\approx} \Phi'_\lambda$ for the statement that $\lim_{\lambda \to \infty} \lambda^n |\Phi_\lambda - \Phi'_\lambda| = 0$ holds for all $n \in \mathbb{N}$.

AL0. Given any sequence of open subsets $\mathcal{O}_\lambda$ of $\mathbb{R}^4$, $B_\lambda \equiv 1 \nearrow D$ is asymptotically localised in $\mathcal{O}_\lambda$.

AL1. Let $(\Lambda, a) \in \mathcal{P}_+^\uparrow$ and let B_λ be asymptotically localised in $\mathcal{O}_\lambda$. Then $B'_\lambda \equiv U(\Lambda, a) B_\lambda U(\Lambda, a)^{-1}$ is asymptotically localised in $\hat{\mathcal{O}}_\lambda \equiv \Lambda \mathcal{O}_\lambda + a$.

AL2. If $B_{j,\lambda}$ is asymptotically localised in $\mathcal{O}_{j,\lambda}$ for $j = 1, 2$, then $B_\lambda \equiv B_{1,\lambda} B_{2,\lambda}$ as well as $B'_\lambda \equiv B_{1,\lambda} + B_{2,\lambda}$ is asymptotically localised in $\mathcal{O}_\lambda \equiv \mathcal{O}_{1,\lambda} \cup \mathcal{O}_{2,\lambda}$.

AL3. If B_λ is asymptotically localised in $\mathcal{O}_\lambda$, then so is $B_\lambda^* \nearrow D$.

AL4. If B_λ is asymptotically localised in $\mathcal{O}_\lambda$, then so is $B'_\lambda \equiv P(\lambda) B_\lambda$ for every polynomial $P(\lambda)$.

AL5. If B_λ is asymptotically localised in $\mathcal{O}_\lambda$, then $B'_\lambda \equiv B_{g(\lambda)}$ is asymptotically localised in $\mathcal{O}'_\lambda \equiv \mathcal{O}_{g(\lambda)}$ for every positive function $g(\lambda)$ fulfilling

$$\lim_{\lambda \to \infty} \lambda^{-n} g(\lambda) = \lim_{\lambda \to \infty} \lambda^{1/n} / g(\lambda) = 0 \qquad \text{for some } n \in \mathbb{N}.$$

AL6. Let $B_{j,\lambda}$ be asymptotically localised in $\mathcal{O}_{j,\lambda}$ for $j = 1, \ldots, n$ and let $\Phi \in D$; then $\|B_{1,\alpha_1} \ldots B_{n,\lambda_n} \Phi\|$ is polynomially bounded in $\lambda_1, \ldots, \lambda_n$ simultaneously for sufficiently large $\lambda_1, \ldots, \lambda_n$.

For arbitrary $B \in \hat{\mathcal{P}}(\mathbb{R}^4)$ and $\varphi \in \mathcal{S}(\mathbb{R}^4)$ we denote by $B(\varphi)$ the unique element of $\hat{\mathcal{P}}(\mathbb{R}^4)$ fulfilling

$$B(\varphi)\Phi = \int \mathrm{d}x\, \varphi(x) U(1, x) B U(1, x)^{-1} \Phi$$

for all $\Phi \in D$. Then, by arguments similar to those used in the proof of lemma 1, we also get

AL7. Let $B \in \hat{\mathcal{P}}(\mathbb{R}^4)$, let $\varepsilon > 0$ and let the $\mathcal{S}(\mathbb{R}^4)$ functions φ_λ be asymptotically localised in $\mathcal{O}_\lambda$. Then the $\hat{\mathcal{P}}(\mathbb{R}^4)$ operators $B(\varphi_\lambda)$ are asymptotically localised in $U_{\varepsilon\lambda}(\mathcal{O}_\lambda)$.

4. Haag–Ruelle–Hepp scattering states

As we shall see below (lemma 3 and proof of lemma 5), the scattering states of the Haag–Ruelle–Hepp theory (Hepp 1964, 1966) are constructed by means of operator sequences of the following type.

Definition 3. Let K be a closed cone, and let Σ be a space-like hyperplane in $\mathbb{R}^4$. A sequence $\{B_\lambda\}_{\lambda > 0} \subset \hat{\mathcal{P}}(\mathbb{R}^4)$ is called a (K, Σ) sequence, iff the following three conditions are fulfilled.

(i) For every $\varepsilon > 0$, B_λ is asymptotically localised in $U_{\varepsilon\lambda}(K \cap \lambda\Sigma)$.

(ii) There is a vector $\Phi \in \mathcal{H}$ with $B_\lambda \Omega \underset{\lambda \to \infty}{\approx} \Phi$.

(iii) $B_\lambda^* B_\lambda \Omega \underset{\lambda \to \infty}{\approx} \langle \Omega | B_\lambda^* B_\lambda \Omega \rangle \Omega$.

Lemma 3. Let $K_1, \ldots, K_n$ be closed cones with $K_j \cap K_k = \{0\}$ for different $j, k \in \{1, \ldots, n\}$, let Σ be a space-like hyperplane not containing the origin and let $\{B_{j,\lambda}\}_{\lambda > 0}$ be a (K_j, Σ) sequence for $j = q, \ldots, n$. Then $\{B_\lambda \equiv B_{1,\lambda} \ldots B_{n,\lambda}\}_{\lambda > 0}$ is a $(K_1 \cup \ldots \cup K_n, \Sigma)$ sequence.

Proof. Obviously, it is sufficient to prove the lemma for the special case $n = 2$: condition (i) of definition 3 is a direct consequence of AL2. In order to prove condition (ii) let us choose some $\varepsilon > 0$, small enough to guarantee $K_1 \cap (1+\varepsilon_1)\Sigma$ to be space-like with respect to $K_2 \cap (1+\varepsilon_2)\Sigma$ for all $\varepsilon_1, \varepsilon_2 \in [0, \varepsilon)$. Moreover, let us choose a sequence $s_\lambda \in [\lambda, (1+\varepsilon)\lambda)$ such that

$$\|B_{1,\lambda}B_{2,\lambda}\Omega - B_{1,s_\lambda}B_{2,s_\lambda}\Omega\| \underset{\lambda\to\infty}{\approx} \sup_{\lambda' \in [\lambda,(1+\varepsilon)\lambda)} \|B_{1,\lambda}B_{2,\lambda}\Omega - B_{1,\lambda'}B_{2,\lambda'}\Omega\|.$$

Since condition (ii) of definition 3 is fulfilled for $B_{2,\lambda}$, we conclude with AL3, AL6 and Schwarz's inequality that

$$B_{1,s_\lambda}B_{2,s_\lambda}\Omega \underset{\lambda\to\infty}{\approx} B_{1,s_\lambda}B_{2,\lambda}\Omega.$$

By AL5 and the corollary we see that

$$B_{1,s_\lambda}B_{2,\lambda}\Omega \underset{\lambda\to\infty}{\approx} B_{2,\lambda}B_{1,s_\lambda}\Omega.$$

Repeating these arguments, we get

$$B_{2,\lambda}B_{1,s_\lambda}\Omega \underset{\lambda\to\infty}{\approx} B_{2,\lambda}B_{1,\lambda}\Omega$$

and

$$B_{2,\lambda}B_{1,\lambda}\Omega \underset{\lambda\to\infty}{\approx} B_{1,\lambda}B_{2,\lambda}\Omega.$$

Putting all the pieces together we get

$$\sup_{\lambda' \in [\lambda,(1+\varepsilon)\lambda)} \|B_{1,\lambda}B_{2,\lambda}\Omega - B_{1,\lambda'}B_{2,\lambda'}\Omega\| \underset{\lambda\to\infty}{\approx} 0$$

which implies condition (ii) of definition 3.

In order to prove (iii), note that, by AL3 and the corollary, we have

$$(B_{1,\lambda}B_{2,\lambda})^* B_{1,\lambda}B_{2,\lambda}\Omega \underset{\lambda\to\infty}{\approx} B^*_{1,\lambda}B_{1,\lambda}B^*_{2,\lambda}B_{2,\lambda}\Omega.$$

Since condition (iii) of definition 3 is fulfilled for $B_{1,\lambda}$ and $B_{2,\lambda}$, we see by AL3, AL6 and Schwarz's inequality that

$$B^*_{1,\lambda}B_{1,\lambda}B^*_{2,\lambda}B_{2,\lambda}\Omega \underset{\lambda\to\infty}{\approx} \langle\Omega|B^*_{1,\lambda}B_{1,\lambda}\Omega\rangle\langle\Omega|B^*_{2,\lambda}B_{2,\lambda}\Omega\rangle\Omega.$$

Hence

$$(B_{1,\lambda}B_{2,\lambda})^* B_{1,\lambda}B_{2,\lambda}\Omega \underset{\lambda\to\infty}{\approx} \langle\Omega|B^*_{1,\lambda}B_{1,\lambda}\Omega\rangle\langle\Omega|B^*_{2,\lambda}B_{2,\lambda}\Omega\rangle\Omega,$$

which implies condition (iii) of definition 3 for $B_\lambda \equiv B_{1,\lambda}B_{2,\lambda}$.

Definition 4. Let $\Phi \in \mathscr{H}$ and let K be a closed cone. Then we say that Φ is *K-approachable*, iff for every space-like hyperplane Σ with $\Sigma \cap K \not\subset \{0\}$ there is a (K, Σ) sequence $\{B_\lambda\}_{\lambda>0}$ with $\Phi = \lim_{\lambda\to\infty} B_\lambda\Omega$.

Let us note that W7 has the following simple consequence (cf proof of relation (22) below):

Lemma 4. Let Δ be a closed subset of $\mathbb{R}^4$ and let $\hat{\Phi} \in D \cap E(\Delta)D$. Then†

$$B(\varphi)\hat{\Phi} \in E(\Delta + \operatorname{supp} \tilde{\varphi})D$$

for all $B \in \hat{\mathscr{P}}(\mathbb{R}^4)$ and all $\varphi \in \mathscr{S}(\mathbb{R}^4)$.

Now, the basis of our scattering formalism is the following.

Lemma 5. Let $M \subset M_m$ and let Φ be a one-particle state vector of the form $\Phi = B\Omega \in E(M)D$, with $B \in \hat{\mathscr{P}}(\mathbb{R}^4)$. If K is a closed cone with apex at the origin, containing $M \cup -M$ in its interior, then Φ is K-approachable.

Proof. Let Σ be an arbitrary space-like hyperplane intersecting the open future (respectively past) light cone. Then we may choose a positive number r and a restricted Lorentz transformation Λ with

$$r\Lambda\Sigma = \Sigma_\pm \equiv \{x \in \mathbb{R}^4 : \pm x^0 = 1\}. \tag{10}$$

If n, k are arbitrary non-negative integers with $n \leqslant k$, then every $\tilde{\psi} \in \mathscr{S}(\mathbb{R}^k)$ may be written in the form

$$\tilde{\psi}(\chi^1, \ldots, \chi^k) = \int d\eta^1 \ldots d\eta^n \, \tilde{h}(\chi^1 - \eta^1, \ldots, \chi^n - \eta^n, \chi^{n+1}, \ldots, \chi^k)\tilde{g}(\eta^1, \ldots, \eta^n)$$

with $\tilde{h} \in \mathscr{S}(\mathbb{R}^k)$, $\tilde{g} \in \mathscr{S}(\mathbb{R}^n)$ and $\int d\eta^1 \ldots d\eta^n \, \tilde{g}(\eta^1, \ldots, \eta^n) = 1$ (Lücke 1974, lemma 1). Hence, by W7 and invariance of Ω, the vector $\Phi' \equiv U(\Lambda, 0)\Phi$ may be written as $\Phi' = \hat{B}(\varphi_0^\pm)\Omega$ with $\hat{B} \in \hat{\mathscr{P}}(\mathbb{R}^4)$ and $\varphi_\lambda^\pm$ of the form (9). Applying lemma 4 to the special case $\hat{\Phi} = \Omega$, $\Delta = \{0\}$ and recalling (2) as well as $\Phi' \in E(\Lambda M)D$, we see that we may choose φ such that

$$\operatorname{supp} \tilde{\varphi} \subset \{p \in \mathbb{R}^4 : p^0 > 0, \tfrac{1}{4}m^2 < p^2 < \tfrac{9}{4}m^2\} \tag{11}$$

and f such that

$$K_f \subset \Lambda \underset{\sim}{K}. \tag{12}$$

Here, as usual, $\underset{\sim}{K}$ denotes the interior of K. Having made such a choice, one can easily check that (11) implies $\widetilde{\varphi_\lambda^\pm}(p) = \widetilde{\varphi_0^\pm}(p)$ for all $\lambda > 0$ and all $p \in \{0\} \cup M_m \cup V_m$. Hence, by (2) and lemma 4, again, we have

$$U(\Lambda, 0)\Phi = \hat{B}(\varphi_\lambda^\pm)\Omega \qquad \text{for all } \lambda > 0. \tag{13}$$

Moreover, applying lemma 4 to the special case $\hat{\Phi} = \Phi'$, $\Delta = \Lambda M$ shows that

$$\hat{B}(\varphi_\lambda^\pm)^* \hat{B}(\varphi_\lambda^\pm)\Omega \in E(\{p \in \mathbb{R}^4 : p^2 < m^2\})D$$

and therefore, by (2) and (4),

$$\hat{B}(\varphi_\lambda^\pm)^* \hat{B}(\varphi_\lambda^\pm)\Omega \in \mathscr{H}_0^0 \qquad \text{for all } \lambda > 0.$$

† As usual, we denote by $\tilde{\varphi}$ the Fourier transform

$$\tilde{\varphi}(p) \equiv (2\pi)^{-2} \int dx \, \varphi(x) \exp(ipx)$$

of φ and by $\operatorname{supp} \tilde{\varphi}$ the support of the function $\tilde{\varphi}$:

$$\operatorname{supp} \tilde{\varphi} \equiv \overline{\{p \in \mathbb{R}^4 : \tilde{\varphi}(p) \neq 0\}}.$$

With (12), lemma 1, AL7 and (13) we conclude that $\hat{B}_\lambda \equiv \hat{B}(\varphi^\pm_\lambda)$ is a $(\Lambda K, \Sigma_\pm)$ sequence. By (10), AL1, AL5 and the invariance of Ω we see that $B_\lambda \equiv U(\Lambda, 0)^{-1}\hat{B}(\varphi^\pm_{\lambda/r})U(\Lambda, 0)$ is a (K, Σ) sequence with $\lim_{\lambda\to\infty} B_\lambda\Omega = \Phi$. Since Σ was allowed to be any space-like hyperplane with $K \cap \Sigma \not\subset \{0\}$, this proves the lemma.

The K-approachable states form a subset of the linear manifold $\mathscr{L}$ defined as follows.

Definition 5. By $\mathscr{L}$ we denote the set of all $\Phi \in \mathscr{H}$ for which there exists a sequence of operators $B_\lambda \in \hat{\mathscr{P}}(\mathbb{R}^4)$, asymptotically localised in $U_\lambda(0)$, with $B_\lambda\Omega \underset{\lambda\to\infty}{\approx} \Phi$.

Without loss of generality we may assume

$$\mathscr{L} \subset D \tag{14}$$

since the field $A(x)$ could be extended to the linear span of $\mathscr{L} \cup D$ otherwise. Then, as a simple consequence of AL3, AL6, Schwarz's inequality and the corollary, we have the following lemma, indicating interesting localisation properties of K-approachable states (cf Knight 1961).

Lemma 6. Let K be a closed cone, let $\Phi \in \mathscr{H}$ be K-approachable, let Σ be a space-like plane with $\Sigma \cap K \not\subset \{0\}$, and let $\mathcal{O}$ be space-like with respect to some neighbourhood of $\Sigma \cap K$. Then

$$\langle\Phi|A_\lambda\Phi\rangle \underset{\lambda\to\infty}{\approx} \langle\Phi|\Phi\rangle\langle\Omega|A_\lambda\Omega\rangle$$

holds for every sequence of $\hat{\mathscr{P}}(\mathbb{R}^4)$ operators A_λ asymptotically localised in $\lambda\mathcal{O}$.

5. Asymptotic condition and its evaluation

We are now in the position to give a rigorous mathematical formulation of the physical characterisation of $\Phi_1 \otimes_\pm \Phi_2$, described qualitatively in § 2.

Asymptotic condition. Let K_1, K_2 be closed cones contained in the closed future (respectively past) light cone with $K_1 \cap K_2 = \{0\}$, let Σ be a space-like hyperplane with $K_j \cap \Sigma \not\subset \{0\}$ for $j = 1, 2$, and let $\mathcal{O}$ be an open subset of $\mathbb{R}^4$ space-like with respect to some neighbourhood of $K_2 \cap \Sigma$. Finally, let $\Phi_j \in V_{\text{out (in)}}\mathscr{H}^0$ be K_j-approachable for $j = 1, 2$. Then $\Phi_1 \otimes_\pm \Phi_2$ should fulfil the following two conditions.

AC1. $\quad \Phi_1 \otimes_\pm \Phi_2 \in D.$

AC2. $\quad \langle\Phi_1 \otimes_\pm \Phi_2|A_\lambda\Phi_1 \otimes_\pm \Phi_2\rangle \underset{\lambda\to\infty}{\approx} \langle\Phi_1|A_\lambda\Phi_1\rangle\langle\Phi_2|\Phi_2\rangle$

if the $\hat{\mathscr{P}}(\mathbb{R}^4)$ operators A_λ are asymptotically localised in $\lambda\mathcal{O}$.

Remarks.

(i) The technical assumption AC1 could be circumvented by the use of bounded observables (cf Araki and Haag 1967).

(ii) Condition AC2 is physically motivated only for those A_λ which are restrictions of self-adjoint operators to D. Actually, we need not worry about that, since we shall use AC2 only for sequences of positive symmetric operators A_λ, which are known to have self-adjoint extensions (Friedrichs 1934).

(iii) Putting $A_\lambda \equiv 1 \diagup D$ in AC2 and recalling AL0 we get $\|\Phi_1 \otimes_\pm \Phi_2\| = \|\Phi_1\| \cdot \|\Phi_2\|$ directly.

According to (6) and (7) we have

$$\Omega \otimes_\pm \Phi = \Phi \otimes_\pm \Omega = \Phi \qquad \text{for all } \Phi \in V_{\text{out (in)}}\mathscr{H}^0. \tag{15}$$

Thus, by lemma 6, the asymptotic condition is fulfilled for the special case $\Phi_1 = \Omega$ (or $\Phi_2 = \Omega$). If neither Φ_1 nor Φ_2 is a multiple of Ω, the asymptotic condition will be used to determine $\Phi_1 \otimes_\pm \Phi_2$.

Theorem. There is a unique isometric mapping V_{out} (respectively V_{in}) from $\mathscr{H}^0$ into $\mathscr{H}$ fulfilling (1) and (6) and such that the asymptotic condition holds for the product $\otimes_+$ (respectively $\otimes_-$) defined by (7). For this product $\otimes_\pm$ the following statement is true: If $K_1, K_2, \Sigma, \Phi_1, \Phi_2$ are as considered in the asymptotic condition, and if $\{B_{j,\lambda}\}_{\lambda>0}$ are (K_j, Σ) sequences with

$$\Phi_j = \lim_{\lambda\to\infty} B_{j,\lambda}\Omega \qquad \text{for } j = 1, 2 \tag{16}$$

then

$$\Phi_1 \otimes_\pm \Phi_2 \underset{\lambda\to\infty}{\approx} B_{1,\lambda}B_{2,\lambda}\Omega. \tag{17}$$

Proof. Let $\mathscr{N}_n^0$ denote the set of all n-particle state vectors of the form $\Phi^0 = \Phi_1^0 \otimes_s \ldots \otimes_s \Phi_n^0$ for which there are pair-wise disjoint closed subsets $\Delta_1, \ldots, \Delta_n$ of M_m with

$$\Phi_j^0 \in (E(\Delta_j)D) \cap \mathscr{P}(\mathbb{R}^4)\Omega$$

and

$$\sup_{p,p'\in\Delta_j} \|p - p'\| < m/5n.$$

We shall prove the theorem in six steps. In the first step we assume existence of an isometric mapping $V_{\text{out (in)}}$ from $\mathscr{H}^0$ into $\mathscr{H}$ fulfilling (1), (6) and the asymptotic condition. We then derive

$$V_{\text{out (in)}}\Phi^0 = \lim_{\lambda\to\infty} B_{1,\lambda}^0 \ldots B_{n,\lambda}^0\Omega \tag{18}$$

for $n \in \mathbb{N}$ and $\Phi^0 \in \mathscr{N}_n^0$, where the $\{B_{j,\lambda}^0\}_{\lambda>0}$ are $(K_j^0, \Sigma_\pm)$ sequences according to lemma 5 with

$$\Phi_j^0 = \lim_{\lambda\to\infty} B_{j,\lambda}^0\Omega$$

and

$$K_j^0 \cap K_k^0 = \{0\} \qquad \text{for } j \neq k.$$

Since $\mathscr{H}_0^0 \cup (\bigcup_{n\in\mathbb{N}} \mathscr{N}_n^0)$ is total in $\mathscr{H}^0$, this determines $V_{\text{out(in)}}$ completely. In the second step, we prove the existence of an isometric operator fulfilling (6) and (18). In the third step, every isometric operator $V_{\text{out (in)}}$ fulfilling (6) and (18) is shown to fulfil (17) for $\Sigma = \Sigma_\pm$. In the fourth step, we prove the right-hand side in (17) to be independent of the special choice for Σ and the sequences $\{B_{j,\lambda}\}_{\lambda>0}$. In the fifth step we show that every *isometric* operator $V_{\mathrm{out\,(in)}}$ fulfilling (6) and (17) fulfils (1). Finally, in the sixth step, we prove that $\otimes_\pm$, as defined by (7), fulfils the asymptotic condition if $V_{\mathrm{out\,(in)}}$ fulfils (17).

Step 1. Let $V_{\mathrm{out\,(in)}}$ be an isometric mapping from $\mathscr{H}^0$ into $\mathscr{H}$ fulfilling (1), (6) and the asymptotic condition. Let K_j, Σ, Φ_j be as required for (17). In order to prove (18), it is clearly sufficient, by lemmas 3 and 4, to show (17) for the special case that there are four-momenta p_j with

$$\Phi_j \in E(U_{m/5}(p_j))D \tag{19}$$

for $j = 1, 2$.

Choose some $\delta \in (0, m/5)$ and test functions $\varphi_j \in \mathscr{S}(\mathbb{R}^4)$ with

$$\operatorname{supp} \tilde{\varphi}_j \subset U_{\delta+m/5}(p_j) \tag{20}$$

and

$$\tilde{\varphi}_j(p) = (2\pi)^{-2} \qquad \text{for } p \in U_{m/5}(p_j).$$

Then, because

$$\int \mathrm{d}x\, \varphi_j(x) U(x) = (2\pi)^2 \int \mathrm{d}E(p)\tilde{\varphi}(p),$$

we have

$$\int \mathrm{d}x\, \varphi_j(x) U(x)\Phi_j = \Phi_j. \tag{21}$$

By (1) and (7) we get

$$U(\Lambda, a)(\Phi_1' \otimes_\pm \Phi_2') = (U(\Lambda, a)\Phi_1) \otimes_\pm (U(\Lambda, a)\Phi_2)$$

for all $(\Lambda, a) \in \mathscr{P}_+^\uparrow$ and all $\Phi_1', \Phi_2' \in V_{\mathrm{out\,(in)}}\mathscr{H}^0$. Therefore (21) implies

$$\begin{aligned}
(2\pi)^2 \int & \mathrm{d}E(p)\tilde{\varphi}(p)(\Phi_1 \otimes_\pm \Phi_2) \\
&= \int \mathrm{d}x\, \mathrm{d}x_1\, \mathrm{d}x_2\, \varphi(x)\varphi_1(x_1)\varphi_2(x_2) U(x)((U(x_1)\Phi_1) \otimes_\pm (U(x_2)\Phi_2)) \\
&= \int \mathrm{d}x\, \mathrm{d}x_1\, \mathrm{d}x_2\, \varphi(x)\varphi_1(x_1)\varphi_2(x_2)(U(x_1+x)\Phi_1) \otimes_\pm (U(x_2+x)\Phi_2) \\
&= \int \mathrm{d}x_1\, \mathrm{d}x_2 \left(\int \mathrm{d}x\, \varphi(x)\varphi_1(x_1-x)\varphi_2(x_2-x)\right)(U(x_1)\Phi_1) \otimes_\pm (U(x_2)\Phi_2)
\end{aligned}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^4)$. Since

$$\int \mathrm{d}x\, \varphi(x)\varphi_1(x_1-x)\varphi_2(x_2-x) = 0$$

if $\operatorname{supp} \tilde{\varphi} \cap (\operatorname{supp} \tilde{\varphi}_1 + \operatorname{supp} \tilde{\varphi}_2) = \varnothing$, and since δ could be chosen arbitrarily small, we conclude with (20) that

$$\Phi_1 \otimes_\pm \Phi_2 \in E(\overline{U_{m/5}(p_1) + U_{m/5}(p_2)})\mathscr{H}. \tag{22}$$

By W7 and invariance of Ω we have

$$B_{j,\lambda}(\varphi_j)\Omega = \int \mathrm{d}x\, \varphi_j(x)U(x)B_{j,\lambda}\Omega$$

for $j = 1, 2$. Therefore, since $\int \mathrm{d}x\, \varphi_j(x)U(x)$ is a bounded operator, we may conclude with (16) and (21) that

$$B_{j,\lambda}(\varphi_j)\Omega \underset{\lambda\to\infty}{\approx} \Phi_j \tag{23}$$

for $j = 1, 2$. Applying lemma 4 twice, we get

$$\begin{aligned} B_{j,\lambda}(\varphi_j)^* B_{j,\lambda}(\varphi_j)\Omega &= B^*_{j,\lambda}(\varphi_j^*)B_{j,\lambda}(\varphi_j)\Omega \\ &\in E(\operatorname{supp}\tilde{\varphi}_j - \operatorname{supp}\tilde{\varphi}_j)\mathscr{H}. \end{aligned}$$

By (20) and the spectrum condition (2), this implies

$$B_{j,\lambda}(\varphi_j)^* B_{j,\lambda}(\varphi_j)\Omega = \langle\Omega|B_{j,\lambda}(\varphi_j)^* B_{j,\lambda}(\varphi_j)\Omega\rangle\Omega. \tag{24}$$

Recalling (22), by similar arguments we get

$$B_{1,\lambda}(\varphi_1)^* B_{2,\lambda}(\varphi_2)^* \Phi_1 \otimes_\pm \Phi_2 = \langle\Omega|B_{1,\lambda}(\varphi_1)^* B_{2,\lambda}(\varphi_2)^* \Phi_1 \otimes_\pm \Phi_2\rangle\Omega. \tag{25}$$

Let us exclude the trivial case that Φ_1 or Φ_2 is a multiple of Ω, in which (17) and (15) are clearly equivalent. Then (19) and the spectrum condition imply $p_1, p_2 \in U_{m/4}(V_+ \backslash U_{m/2}(0))$, V_+ denoting the open future light cone. By (20), lemma 4 and (2), this implies

$$B_{j,\lambda}(\varphi_j)^*\Omega = B^*_{j,\lambda}(\varphi_j^*)\Omega = 0 \tag{26}$$

for $j = 1, 2$. Now, let us consider arbitrary complex numbers $\mathfrak{z}_1$, $\mathfrak{z}_2$ and define

$$\Phi_j' \equiv \Phi_j + \mathfrak{z}_j\Omega \tag{27}$$

and

$$B'_{j,\lambda} \equiv B_{j,\lambda}(\varphi_j) + \mathfrak{z}_j\left(1 - \frac{B_{j,\lambda}(\varphi_j)B_{j,\lambda}(\varphi_j)^*}{\langle\Phi_j|\Phi_j\rangle}\right). \tag{28}$$

One can easily prove that, for arbitrary $\varepsilon > 0$, $B'_{j,\lambda}$ is also asymptotically localised in $U_{\varepsilon\lambda}(K_j \cap \lambda\Sigma)$. From (23) and (26) we get

$$B'_{j,\lambda}\Omega \underset{\lambda\to\infty}{\approx} \Phi_j' \qquad \text{for } j = 1, 2. \tag{29}$$

From (23), (24) and (26), on the other hand, we get

$$(B'_{j,\lambda})^* B'_{j,\lambda}\Omega \in \mathscr{H}_0^0 \qquad \text{for all } \lambda > 0 \text{ and } j = 1, 2. \tag{30}$$

Summarising, we see also Φ_j' to be K_j-approachable. Therefore, by AL2 and AL3, we may apply the asymptotic condition to get

$$\begin{aligned} &\left\|\Phi_1' \otimes_\pm \Phi_2' - \frac{B'_{1,\lambda}B'^*_{2,\lambda}}{\langle\Phi_1'|\Phi_1'\rangle}\Phi_1' \otimes_\pm \Phi_2'\right\|^2 \\ &\quad \underset{\lambda\to\infty}{\approx} \left[\|\Phi_1\|^2\|\Phi_2\|^2 - 2\left\langle\Phi_1'\left|\frac{B'_{1,\lambda}B'^*_{1,\lambda}}{\langle\Phi_1'|\Phi_1'\rangle}\Phi_1'\right.\right\rangle\langle\Phi_2'|\Phi_2'\rangle\right. \\ &\qquad \left. + 2\left\langle\Phi_1'\left|\left(\frac{B'_{1,\lambda}B'^*_{1,\lambda}}{\langle\Phi_1'|\Phi_1'\rangle}\right)^2\Phi_1'\right.\right\rangle\langle\Phi_2'|\Phi_2'\rangle\right]. \end{aligned}$$

By (29), (30), AL3, AL6 and Schwarz's inequality we see the right-hand side to converge rapidly to zero. By symmetry of $\otimes_\pm$ we get a similar result for $B'_{1,\lambda}$ replaced by $B'_{2,\lambda}$. Thus:

$$\frac{B'_{j,\lambda}B'^{*}_{j,\lambda}}{\langle\Phi'_j|\Phi'_j\rangle}\Phi'_1\otimes_\pm\Phi'_2 \underset{\lambda\to\infty}{\approx} \Phi'_1\otimes_\pm\Phi'_2 \qquad \text{for } j=1,2. \tag{31}$$

Hence, by AL3, AL6 and Schwarz's inequality, again, we also have

$$\frac{B'_{1,\lambda}B'^{*}_{1,\lambda}B'_{2,\lambda}B'^{*}_{2,\lambda}}{\langle\Phi'_1|\Phi'_1\rangle\langle\Phi'_2|\Phi'_2\rangle}\Phi'_1\otimes_\pm\Phi'_2 \underset{\lambda\to\infty}{\approx} \Phi'_1\otimes_\pm\Phi'_2. \tag{32}$$

Since, by (15) and (27),

$$\Phi'_1\otimes_\pm\Phi'_2=\Phi_1\otimes_\pm\Phi_2+\mathfrak{z}_2\Phi_1+\mathfrak{z}_1\Phi_2+\mathfrak{z}_1\mathfrak{z}_2\Omega, \tag{33}$$

one can easily check, using (23)–(26), (28), AL3, AL6, Schwarz's inequality and the corollary, that

$$B'^{*}_{1,\lambda}B'^{*}_{2,\lambda}\Phi'_1\otimes_\pm\Phi'_2 \underset{\lambda\to\infty}{\approx} \langle\Omega|B'^{*}_{1,\lambda}B'^{*}_{2,\lambda}\Phi'_1\otimes_\pm\Phi'_2\rangle\Omega.$$

Therefore, by (32), AL3, AL6, Schwarz's inequality, the corollary and lemma 3, we see that there must be a complex number $\rho=\rho(\mathfrak{z}_1,\mathfrak{z}_2)$ with

$$\rho B'_{1,\lambda}B'_{2,\lambda}\Omega \underset{\lambda\to\infty}{\approx} \Phi'_1\otimes_\pm\Phi'_2. \tag{34}$$

With (23), (27) and (28) this implies

$$\begin{aligned}\rho(0,0)(\Phi_1+\Phi_1\otimes_\pm\Phi_2)&=\rho(0,0)\Phi_1\otimes_\pm(\Phi_2+\Omega)\\&=\rho(0,0)\rho(0,1)\lim_{\lambda\to\infty}B_{1,\lambda}(\varphi_1)(1+B_{2,\lambda}(\varphi_2))\Omega\\&=\rho(0,0)\rho(0,1)\Phi_1+\rho(0,1)\Phi_1\otimes_\pm\Phi_2.\end{aligned} \tag{35}$$

By (34), (23), (26) and the corollary, we see that $\langle\Phi_1|\Phi_1\otimes_\pm\Phi_2\rangle=0$, hence (35) implies $\rho(0,1)=\rho(0,0)=1$. Since, applying now familiar arguments, we have

$$\begin{aligned}B_{1,\lambda}(\varphi_1)B_{2,\lambda}(\varphi_2)\Omega &\underset{\lambda\to\infty}{\approx} B_{1,\lambda}(\varphi_1)B_{2,\lambda}\Omega\\&\underset{\lambda\to\infty}{\approx} B_{2,\lambda}B_{1,\lambda}(\varphi_1)\Omega\\&\underset{\lambda\to\infty}{\approx} B_{2,\lambda}B_{1,\lambda}\Omega\\&\underset{\lambda\to\infty}{\approx} B_{1,\lambda}B_{2,\lambda}\Omega,\end{aligned}$$

this proves (17) for the special case (19).

Step 2. Using the corollary, one can easily check the right-hand side of (18) to be independent of the special choice of the sequences $\{B^0_{j,\lambda}\}_{\lambda>0}$, existence of the limit being guaranteed by lemma 3. Therefore, by lemma 5, we may define $V_{\text{out (in)}}$ on $\mathcal{N}^0_n$ by (18) for $n=2,3,\ldots$. On the other hand, by well-exercised reasoning, we can easily prove that (18) and (6) imply

$$\langle V_{\text{out (in)}}\Phi^0|V_{\text{out (in)}}\Psi^0\rangle=(\Phi^0|\Psi^0)$$

for $\Phi^0, \Psi^0 \in \mathscr{H}_0^0 \cup \mathscr{H}_1^0 \cup (\bigcup_{n>1} \mathcal{N}_n^0)$, $(\ |\ )$ denoting the inner product of $\mathscr{H}^0$. Therefore $V_{\text{out (in)}}$, as defined by (6) and (18), has an isometric extension to all of $\mathscr{H}^0$.

Step 3. Let $V_{\text{out (in)}}$ be an isometric mapping from $\mathscr{H}^0$ into $\mathscr{H}$ fulfilling (6) and (18). Moreover, let K_j, Σ, Φ_j and $\{B_{j,\lambda}\}_{\lambda>0}$ be as required for (17) and consider the special case $\Sigma = \Sigma_\pm$. We have to prove

$$\|\Phi_1 \otimes_\pm \Phi_2 - \lim_{\lambda\to\infty} B_{1,\lambda} B_{2,\lambda}\Omega\| < \varepsilon \tag{36}$$

for arbitrary $\varepsilon > 0$. Let us first consider the special case that there is a four-momentum p_2 with

$$\Phi_2 \in E(U_{m/5}(p_2))D. \tag{37}$$

Denote by $\mathcal{N}_n^\pm(K)$, $K \subset \mathbb{R}^4$, the set of all

$$\Phi = V_{\text{out (in)}}(\Phi_1^0 \otimes_s \ldots \otimes_s \Phi_n^0) \in V_{\text{out (in)}}\mathscr{H}_n^0$$

with $\Phi_j^0 \in E(M_m \cap (K \cup -K))D$ for $j = 1, \ldots, n$. Using (16), (18) and our standard techniques, we easily see that Φ_j is in the closed linear span of $\mathscr{H}_0^0 \cup (\bigcup_{n\in\mathbb{N}} \mathcal{N}_n^\pm(K_j))$. Therefore, thanks to isometry of $V_{\text{out (in)}}$, we may choose Ψ_1 from the linear span of $\mathscr{H}_0^0 \cup (\bigcup_{n\in\mathbb{N}} \mathcal{N}_n^\pm(K_1))$ and Ψ_2 from the linear span of $\mathscr{H}_0^0 \cup ((\bigcup_{n\in\mathbb{N}} \mathcal{N}_n^\pm(K_2)) \cap E(U_{m/5}(p_2))D)$ such that

$$\|\Psi_1 \otimes_\pm \Psi_2 - \Phi_1 \otimes_\pm \Phi_2\| < \varepsilon/2.$$

Similarly, using (18) (respectively (6)) and isometry of $V_{\text{out (in)}}$ we easily derive

$$\begin{aligned}
&\|\Psi_1 \otimes_\pm \Psi_2 - \lim_{\lambda\to\infty} B_{1,\lambda}B_{2,\lambda}\Omega\|^2 \\
&\quad = \|\Psi_1 \otimes_\pm \Psi_2\|^2 + \langle\Phi_1|\Phi_1\rangle\langle\Phi_2|\Phi_2\rangle - 2\,\mathrm{Re}(\langle\Psi_1|\Phi_1\rangle\langle\Psi_2|\Phi_2\rangle) \\
&\quad = \|\Psi_1 \otimes_\pm \Psi_2 - \Phi_1 \otimes_\pm \Phi_2\|^2 < (\varepsilon/2)^2.
\end{aligned}$$

By the triangle inequality, this implies (36) for the special case (37).

Exploiting the linear dependence of $\Phi_1 \otimes_\pm \Phi_2$ on Φ_2 we easily get (17) for the more general case:

$$\Phi_2 \in E(\Delta)D \qquad \Delta \text{ compact.} \tag{38}$$

The general case, finally, may be reduced to (38) as follows: By lemma 4, we have

$$\lim_{\lambda\to\infty} B_{1,\lambda}B_{2,\lambda}(\varphi_\nu)\Omega = \Phi_1 \otimes_\pm \left(\int \mathrm{d}x\, \varphi_\nu(x) U(x)\Phi_2\right)$$

for $\tilde{\varphi}_\nu \in \mathscr{D}(\mathbb{R}^4)$, since (17) is already proved for the case (38). Therefore, thanks to (7) and isometry of $V_{\text{out (in)}}$, $\lim_{\lambda\to\infty} B_{1,\lambda}B_{2,\lambda}(\varphi_\nu)\Omega$ tends to $\Phi_1 \otimes_\pm \Phi_2$ for $\nu \to \infty$, if $\varphi_\nu(x) \to \delta(x)$ for $\nu \to \infty$. On the other hand, we easily derive in our standard way

$$\|\lim_{\lambda\to\infty} B_{1,\lambda}(B_{2,\lambda} - B_{2,\lambda}(\varphi_\nu))\Omega\|^2 = \|\Phi_1\|^2 \left\|\Phi_2 - \int \mathrm{d}x\, \varphi_\nu(x) U(x)\Phi_2\right\|^2.$$

Thus, in the limit $\nu \to \infty$, we get (17) for the general case with $\Sigma = \Sigma_\pm$.

Step 4. Let K_j, Φ_j, Σ and $\{B_{j,\lambda}\}_{\lambda>0}$ be as required for (17). Let Σ' be another space-like hyperplane intersecting the future (respectively past) light cone, and let $\{B'_{j,\lambda}\}_{\lambda>0}$ be

(K_j, Σ') sequences with

$$\Phi_j = \lim_{\lambda\to\infty} B'_{j,\lambda}\Omega \qquad \text{for } j = 1, 2. \tag{39}$$

We have to show that

$$\lim_{\lambda\to\infty} B_{1,\lambda}B_{2,\lambda}\Omega = \lim_{\lambda\to\infty} B'_{1,\lambda}B'_{2,\lambda}\Omega. \tag{40}$$

Obviously, since the Φ_j are K_j-approachable, it is sufficient to prove (40) under the additional assumptions

$$K_1 \cap \Sigma' \times K_2 \cap \Sigma \quad \text{and} \quad K_1 \cap \Sigma \times K_2 \cap \Sigma'.$$

In this case, we may apply our standard techniques once again.

Due to condition (i) of definition 3 and the corollary we have

$$B^{(\prime)}_{1,\lambda}B'_{2,\lambda}\Omega \underset{\lambda\to\infty}{\approx} B'_{2,\lambda}B^{(\prime)}_{1,\lambda}\Omega.$$

On the other hand, due to AL3, AL6, Schwarz's inequality and condition (ii) of definition 3, we have

$$B_{1,\lambda}B_{2,\lambda}\Omega \underset{\lambda\to\infty}{\approx} B_{1,\lambda}B'_{2,\lambda}\Omega$$

and

$$B'_{2,\lambda}B_{1,\lambda}\Omega \underset{\lambda\to\infty}{\approx} B'_{2,\lambda}B'_{1,\lambda}\Omega.$$

Summarising, we get (40).

Step 5. Let $V_{\text{out (in)}}$ be an isometric mapping from $\mathscr{H}^0$ into $\mathscr{H}$ for which (6) and (17) hold, $\otimes_\pm$ being defined by (7). We have to prove (1). Since by (15) and lemmas 3–5, the set of all $\Phi_1 \otimes_\pm \Phi_2$ with Φ_1, Φ_2 as considered in the asymptotic condition is total in $V_{\text{out (in)}}\mathscr{H}^0$, it is sufficient to show that

$$U(\Lambda, a)(\Phi_1 \otimes_\pm \Phi_2) = (U(\Lambda, a)\Phi_1) \otimes_\pm (U(\Lambda, a)\Phi_2) \tag{41}$$

for such Φ_1, Φ_2 and for all $(\Lambda, a) \in \mathscr{P}^\uparrow_+$.

By assumption, there are cones K_1, K_2 with $K_1 \cap K_2 = \{0\}$ and $(K_j, \Sigma_\pm)$ sequences $\{B_{j,\lambda}\}_{\lambda>0}$ with $\Phi_j = \lim_{\lambda\to\infty} B_{j,\lambda}\Omega$ for $j = 1, 2$. Then, by AL1 and invariance of Ω, the operators $B'_{j,\lambda} \equiv U(\Lambda, a)B_{j,\lambda}U(\Lambda, a)^{-1}$ form $(\Lambda K_j, \Lambda\Sigma_\pm)$ sequences with $\Phi'_j \equiv U(\Lambda, a)\Phi_j = \lim_{\lambda\to\infty} B'_{j,\lambda}\Omega$ for arbitrary $(\Lambda, a) \in \mathscr{P}^\uparrow_+$ and $j = 1, 2$. Thus, we may apply (17) to $K'_j \equiv \Lambda K_j$, $\Sigma' \equiv \Lambda\Sigma$, Φ'_j and $B'_{j,\lambda}$ to obtain

$$(U(\Lambda, a)\Phi_1) \otimes_\pm (U(\Lambda, a)\Phi_2) = \lim_{\lambda\to\infty} U(\Lambda, a)B_{1,\lambda}B_{2,\lambda}\Omega.$$

By (17) and continuity of $U(\Lambda, a)$ this implies (41).

Step 6. Let $\otimes_\pm$ fulfil (17), and let K_j, Σ, $\mathscr{O}$, Φ_j and A_λ be as considered in the asymptotic condition. We have to prove that

$$\langle \Phi_1 \otimes_\pm \Phi_2 | A_\lambda \Phi_1 \otimes_\pm \Phi_2 \rangle \underset{\lambda\to\infty}{\approx} \langle \Phi_1 | A_\lambda \Phi_1 \rangle \langle \Phi_2 | \Phi_2 \rangle. \tag{42}$$

To this end, let us choose (K_j, Σ) sequences $\{B_{j,\lambda}\}_{\lambda>0}$ with $\Phi_j = \lim_{\lambda\to\infty} B_{j,\lambda}\Omega$ and apply standard arguments.

By (17), AL3, AL6 and Schwarz's inequality, we get

$$\langle \Phi_1 \otimes_\pm \Phi_2 | A_\lambda \Phi_1 \otimes_\pm \Phi_2 \rangle \underset{\lambda\to\infty}{\approx} \langle \Omega | B^*_{2,\lambda} B^*_{1,\lambda} A_\lambda B_{1,\lambda} B_{2,\lambda} \Omega \rangle.$$

By AL3 and the corollary, this gives

$$\langle \Phi_1 \otimes_\pm \Phi_2 | A_\lambda \Phi_1 \otimes_\pm \Phi_2 \rangle \underset{\lambda\to\infty}{\approx} \langle \Omega | B^*_{1,\lambda} A_\lambda B_{1,\lambda} B^*_{2,\lambda} B_{2,\lambda} \Omega \rangle.$$

Exploiting condition (iii) of definition 3 for $B_{2,\lambda}$ in connection with AL3, AL6 and Schwarz's inequality, we conclude:

$$\langle \Phi_1 \otimes_\pm \Phi_2 | A_\lambda \Phi_1 \otimes_\pm \Phi_2 \rangle \underset{\lambda\to\infty}{\approx} \langle \Omega | B^*_{1,\lambda} A_\lambda B_{1,\lambda} \Omega \rangle \langle \Omega | B^*_{2,\lambda} B_{2,\lambda} \Omega \rangle.$$

Recalling $B_{j,\lambda}\Omega \underset{\lambda\to\infty}{\approx} \Phi_j$, we finally get (42) by AL3, AL6 and Schwarz's inequality, again.

Needless to say, iterating (17) we get the Haag–Ruelle–Hepp scattering theory.

6. Conclusions

Our results can be summarised as follows: Consider a relativistic quantum theory on the Hilbert space $\mathscr{H}$ with continuous unitary representation $U(\Lambda, a)$ of $\mathscr{P}^\uparrow_+$ fulfilling the spectrum condition (2). Let $\hat{\mathscr{P}}(\mathbb{R}^4)$ be a * algebra of operators in $\mathscr{H}$ with respect to which $E(\{0\})\mathscr{H}$ is cyclic. Then a scattering theory is fixed, via the asymptotic condition of § 5, by any notion of asymptotic localisation of sequences of $\hat{\mathscr{P}}(\mathbb{R}^4)$ operators, fulfilling the corollary and AL0–AL7.

In a sloppy way, we may conclude that the S-matrix of a relativistic quantum theory with short-range interaction depends only on the rough localisation properties of observables.

Given $\mathscr{H}$ and $U(\Lambda, a)$, we may always choose a notion of asymptotic localisation, in agreement with the corollary and AL0–AL7, for which the S-matrix becomes trivial. Thus, the crucial question is to find a *physically justified* notion of asymptotic localisation. The use of a concrete field theory is just to provide this information.

A similar view on field theory was expressed by Haag (1970), *relying on* the physical relevance of the Haag–Ruelle scattering formalism. The point of the present paper is that we *derived* this formalism from considerations concerning localisation properties of observables and states.

The analysis could have been simplified to some extent by working with bounded observables. Working with smeared field operators, however, has the advantage that the methods are also applicable to non-localisable fields (Lücke 1978). Let us finally note that the methods are also easily adjustable to non-relativistic quantum field theories (cf Hepp 1965, Sandhas 1966).

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